CRACK ZONE AND CRACK FRONT IN AN ELASTIC BODY UNDER PRESSURE

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In [1] a description was given of the crack zone and crack front in a brittle elastic body under high pressure at the wall of a cavity inside the body. In the present paper, this description is analyzed and an approximate solution of the problem of the propagation of the crack front and the motion of the medium is proposed.

§1. The equation of spherically symmetric motion of an elastic medium has the form

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_r}{\partial r} + \frac{2(\sigma_r - \sigma_\theta)}{r}. \tag{1.1}$$

Here r and t are the coordinate and time, respectively, u is the displacement of a particle of the material, ρ is density, and σ_r and σ_θ are the radial and azimuthal stresses, respectively. For spherically symmetric motion of a medium in which azimuthal stresses are absent due to the presence of radial cracks, the equation of motion has the form

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_r}{\partial r} + 2\frac{\sigma_r}{r}. \tag{1.2}$$

From equations (1.1 and 1.2), with the aid of Hooke's law, we correspondingly obtain

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right), \quad c^2 = \frac{E(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)}, \tag{1.3}$$

$$\frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}, \qquad c_0^2 = \frac{E}{\rho}, \tag{1.4}$$

where σ is Poisson's ratio, and E is Young's modulus. The general solution of equation (1.3) has the form

$$u = -\frac{\partial}{\partial r}\left(\frac{\psi_1 + \psi_2}{r}\right), \quad \psi_1 = \psi_1\left(t - \frac{r - r_0}{c}\right),$$
$$\psi_2 = \psi_2\left(t + \frac{r - r_0}{c}\right), \tag{1.5}$$

where ψ_1 and ψ_2 are arbitrary functions, and r_0 is the initial radius of the cavity, which we introduce for convenience in the following analysis. The solution of equation (1.4) is as follows:

$$u = \frac{f_1 + f_2}{r}, \qquad f_1 = f_1\left(t - \frac{r - r_0}{c_0}\right),$$
$$f_2 = f_2\left(t + \frac{r - r_0}{c_0}\right), \tag{1.6}$$

where f_1 and f_2 are arbitrary functions which, like ψ_1 and ψ_2, are found from the boundary conditions. For a continuous elastic medium

$$v = \frac{\partial u}{\partial t}, \qquad \sigma_r = \frac{E[(1-\sigma)u_{rr} + 2\sigma u_{\theta\theta}]}{(1+\sigma)(1-2\sigma)},$$
$$\sigma_\theta = \sigma_\varphi = \frac{E(u_{\theta\theta} + \sigma u_{rr})}{(1+\sigma)(1-2\sigma)}, \quad u_{rr} = \frac{\partial u}{\partial r}, \quad u_{\theta\theta} = \frac{u}{r}. \tag{1.7}$$

For the radial crack zone

$$v = \frac{\partial u}{\partial t}, \qquad \sigma_r = E\frac{\partial u}{\partial r}. \tag{1.8}$$

As in [1], we examine the following problem. At a certain moment of time, on a sphere of radius r_0 within a solid body, a pressure is initiated that generates a spherical elastic wave. The tensile stresses created in this wave lead to the formation of a radial crack zone in which azimuthal stresses are absent.

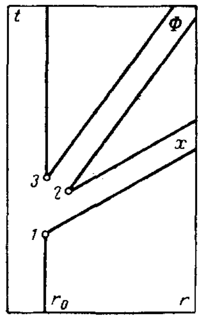

Fig. 1

The dividing line between zones is called the crack front. Since the elastic wave moves in an unperturbed medium at rest, it will be a wave traveling in one direction and described by one function ψ_1. To determine the form of this function, it is sufficient to have a single condition at one boundary, for example, at the cavity wall or the crack front. The motion in the crack region which extends over the perturbed zone is described by the functions f_1 and f_2, whose determination requires conditions at two boundaries: the cavity and the crack front. In [1], a system of equations was given in which the reflected wave (function f_2) was neglected on the basis of conditions that were not explicitly formulated. In this case, the system becomes overdetermined. This approximate approach will be treated in more detail at the end of the paper.

We write the conditions of conservation of mass and momentum at the crack front

$$\rho_2(v_2 - R^{\cdot}) = \rho_1(v_1 - R^{\cdot}),$$
$$\rho_2(v_2 - R^{\cdot})^2 - \sigma_{r2} = \rho_1(v_1 - R^{\cdot})^2 - \sigma_{r1}, \tag{1.9}$$

where $R^{\cdot}$ is the velocity of the front, ρ_1, v_1 and σ_{r1} are the density, mass flow rate, and radial stress in the elastic zone ahead of the crack front, and ρ_2, v_2, and σ_{r2} are the corresponding quantities behind the front in the crack zone. For the density (in the case of spherical symmetry) we have the following expression:

$$\frac{\rho_0}{\rho} = 1 - (1 - 2\sigma)\left(\frac{-\sigma_r - 2\sigma_\theta}{E}\right),$$

where ρ_0 is the initial density of the medium. In correspondence with the concepts of [1], in the crack zone $\sigma_\theta \equiv 0$, and at the crack front $\sigma_\theta = \sigma_{\theta *}$ (critical).

In elasticity, due to the small displacement of the cavity, the boundary condition is usually referred to the initial radius of the cavity, while the pressure at the wall is always taken as independent of the motion in the elastic medium. Such an approximation may, naturally, require adjustments associated with the consideration of the displacement of the boundary or the consideration of the motion of the gas inside the cavity, since it is possible that an abrupt rupture of the elastic medium may lead to the formation of a wave directed toward the center of the cavity, which in its turn may lead to an increase in pressure at the boundary with the elastic medium. The problem becomes more complicated when the medium ceases to be elastic.

We will examine in more detail the possible types of discontinuities at the crack front in the approximation ordinarily used for elasticity. The pressure p(t) is given at the cavity wall. At the moment t_* at which the crack front forms, the pressure remains continuous. Figure 1 gives a schematic representation of the wall, the crack front (Φ), and the characteristic (x), the origin of which is in the point at which the crack front forms. In [1] it was assumed that $\sigma_{\theta *}$ is reached at the cavity wall, and then remains unchanged at the crack front. This corresponds to the assumption that σ_θ is continuous at the characteristic. It should be noted here that a more general, though still relatively simple, assumption would be that at the characteristic σ_θ has a discontinuity, i. e., the formation of the crack front takes place at a certain σ_θ, say larger than that which subsequently exists at the crack front. In our case it is also natural to assume continuity of displacement across the characteristic. From these two assumptions, since at both sides of the characteristic (points 1 and 2 in Fig. 1)

$$-\frac{\sigma_\theta}{\rho c^2} = \frac{\sigma}{(1-\sigma)}\frac{\psi''}{c^2 r} - \frac{(1-2\sigma)}{(1-\sigma)}\frac{u}{r}, \tag{1.11}$$

it follows that ψ'' is also continuous at the characteristic. Then, since at both sides of the characteristic

$$-\frac{\sigma_r}{\rho c^2} = \frac{\psi''}{c^2 r} + \frac{2(1-2\sigma)}{(1-\sigma)}\frac{u}{r}, \tag{1.12}$$

it follows that σ_r is also continuous at the characteristic. From the condition of mass and momentum conservation at the characteristic, it follows that v, ψ, and ψ' are also continuous at the characteristic. In virtue of the continuity of pressure at the cavity wall (at points 1 and 3, since the points 1, 2, and 3 represent one physical point), σ_r is continuous at the crack front (points 2 and 3 in Fig. 1). From (1.9) and (1.10) it can be seen that the continuity of σ_r leads either to the condition

$$v_1 = v_2 = R', \qquad \frac{\rho_0}{\rho_2} = \frac{\rho_0}{\rho_1} - \frac{2\sigma_{\theta *}}{E}, \tag{1.13}$$

in which case, however, a crack front does not form because it moves together with the cavity boundary, or to the condition

$$v_1 = v_2, \qquad \rho_1 = \rho_2, \tag{1.14}$$

i. e., the continuity of all functions at the crack front. It follows from (1.10), however, that this condition can be precisely satisfied only for $\sigma_{\theta *} = 0$. Without analyzing this situation more closely, we will examine the case $\sigma_{\theta *} = 0$. This case is doubtless of practical significance, since numerous fissured materials do not exhibit tensile strength in practice, while for numerous materials $\sigma_{\theta *} \ll E$, which is precisely the requirement for $\rho_1 \approx \rho_2$.

§2. The condition $\sigma_{\theta *} = 0$ and the continuity of the radial stress at the crack front lead to the relations

$$-\frac{\sigma_r}{E}R = \frac{1}{(1-2\sigma)}\frac{\psi''}{c^2} = \frac{1}{\sigma}\left(\frac{\psi'}{cR} + \frac{\psi}{R^2}\right) = \frac{f_1' - f_2'}{c_0} + \frac{f_1 + f_2}{R},$$

$$\psi = \psi\left(t - \frac{R - r_0}{c}\right),\ f_1 = f_1\left(t - \frac{R - r_0}{c_0}\right), \tag{2.1}$$

$$f_2 = f_2\left(t + \frac{R - r_0}{c_0}\right),$$

where R is the radius of the crack front.

The condition of velocity continuity at the front yields

$$vR = \frac{\psi''}{c} + \frac{\psi'}{R} = f_1' + f_2'. \tag{2.2}$$

The condition at the cavity boundary ($r = r_0$) for the crack region has the form

$$\frac{f_{10}' - f_{20}'}{c_0 r_0} + \frac{f_{10} + f_{20}}{r_0^2} = \frac{p(t)}{E},$$

$$f_{10} = f_1(t),\ f_{20} = f_2(t),\ f_{10}' = f_1'(t),\ f_{20}' = f_2'(t). \tag{2.3}$$

It is readily shown that the condition of continuity of displacements at the front

$$f_1 + f_2 = \frac{\psi'}{c} + \frac{\psi}{R} \tag{2.4}$$

does not introduce any thing new. We differentiate (2.4) along R(t):

$$f_1'\left(1 - \frac{R'}{c_0}\right) + f_2'\left(1 + \frac{R'}{c_0}\right) =$$

$$= \left(\frac{\psi''}{c} + \frac{\psi'}{R}\right)\left(1 - \frac{R'}{c}\right) - \frac{\psi}{R^2}R'$$

or

$$f_1' + f_2' - \left(\frac{\psi''}{c} + \frac{\psi'}{R}\right) =$$

$$= R'\left[\frac{f_1' - f_2'}{c_0} - \left(\frac{\psi''}{c^2} + \frac{\psi'}{cR} + \frac{\psi}{R^2}\right)\right].$$

Making use of the relation derived from the condition $\sigma_{\theta *} = 0$ and (2.4), we obtain

$$f_1' + f_2' - \left(\frac{\psi''}{c} + \frac{\psi'}{R}\right) = \tag{2.5}$$

$$= R^{\cdot}\left[\frac{f_1^{\cdot}-f_2^{\cdot}}{c_0}+\frac{f_1+f_2}{R}-\frac{1}{\sigma}\left(\frac{\psi^{\cdot}}{cR}+\frac{\psi}{R^2}\right)\right]. \quad (2.5) \text{ (Cont'd)}$$

It can be seen that (2.5) and (2.1) lead directly to (2.2), while (2.5) and (2.2) lead to the last term of (2.1)—continuity of stress.

Hence, for the four unknown functions R, ψ, f_1, and f_2 we have the four equations

$$\frac{\sigma}{(1-2\sigma)}\frac{\psi^{\cdot\cdot}}{c^2}=\frac{\psi^{\cdot}}{cR}+\frac{\psi}{R^2},\quad \frac{f_1^{\cdot}-f_2^{\cdot}}{c_0}=\frac{(1-\sigma)}{\sigma}\frac{(f_1+f_2)}{R},$$

$$\frac{\psi^{\cdot\cdot}}{c}+\frac{\psi^{\cdot}}{R}=f_1^{\cdot}+f_2^{\cdot},\quad \frac{f_{10}^{\cdot}-f_{20}^{\cdot}}{c_0 r_0}+\frac{f_{10}+f_{20}}{r_0^2}=\frac{p}{E}. \quad (2.6)$$

The initial conditions at the point $R = r_0$, $t = t_*$, and the point itself, are found from the known motion that precedes the formation of the crack zone. The solution for the spherically symmetric motion of an elastic medium for a given law governing the pressure variation at the wall of a spherical cavity has the form

$$\psi(\tau)=\frac{r_0^3(1-\sigma)}{\rho c^2\sqrt{1-2\sigma}}e^{-\alpha\tau}(J_1\sin\beta\tau-J_2\cos\beta\tau),$$

$$\psi^{\cdot}(\tau)=\frac{r_0^2}{\rho c}e^{-\alpha\tau}[J_1(\cos\beta\tau-\sqrt{1-2\sigma}\sin\beta\tau)+$$

$$+J_2(\sin\beta\tau+\sqrt{1-2\sigma}\cos\beta\tau)],$$

$$\psi^{\cdot\cdot}(\tau)=\frac{r_0}{\rho}\frac{2\sqrt{1-2\sigma}}{(1-\sigma)}e^{-\alpha\tau}\Big[-J_1\sqrt{1-2\sigma}\Big(\cos\beta\tau+$$

$$+\frac{\sigma}{\sqrt{1-2\sigma}}\sin\beta\tau\Big)+J_2\sigma\Big(\cos\beta\tau-\frac{\sqrt{1-2\sigma}}{\sigma}\sin\beta\tau\Big)\Big]+\frac{r_0}{\rho}p(\tau),$$

$$J_1=\int_0^\tau p(\tau)e^{\alpha\tau}\cos\beta\tau\,d\tau,\qquad J_2=\int_0^\tau p(\tau)e^{\alpha\tau}\sin\beta\tau\,d\tau,$$

$$\alpha=\frac{(1-2\sigma)}{(1-\sigma)}\frac{c}{r_0},\qquad \beta=\frac{\sqrt{1-2\sigma}}{(1-\sigma)}\frac{c}{r_0}.$$

For the step $p = p_0 = \text{const}$

$$\psi(\tau)=\frac{p_0r_0^3}{\rho c^2}\frac{(1-\sigma)}{2(1-2\sigma)}[1-e^{-\alpha\tau}(\cos\beta\tau+\sqrt{1-2\sigma}\sin\beta\tau)],$$

$$\psi^{\cdot}(\tau)=\frac{p_0r_0^2}{\rho c}\frac{(1-\sigma)}{\sqrt{1-2\sigma}}e^{-\alpha\tau}\sin\beta\tau, \quad (2.7)$$

$$\psi^{\cdot\cdot}(\tau)=\frac{p_0r_0}{\rho}e^{-\alpha\tau}(\cos\beta\tau-\sqrt{1-2\sigma}\sin\beta\tau).$$

It is evident that in this case, the critical tensile stress $\sigma_{\theta *} = 0$ is reached first at the cavity wall. The relation for the parameters at the wall is written in the form

$$\frac{\sigma_{\theta 0}}{p_0}=\frac{1}{2}\left[1-\frac{1+\sigma}{1-\sigma}e^{-\alpha\tau}(\cos\beta\tau-\sqrt{1-2\sigma}\sin\beta t)\right],$$

$$\frac{u_0}{r_0}=\frac{p_0}{\rho c^2}\frac{(1-\sigma)}{2(1-2\sigma)}[1-e^{-\alpha t}(\cos\beta t-\sqrt{1-2\sigma}\sin\beta t)],$$

$$\frac{v_0}{c}=\frac{p_0}{\rho c^2}e^{-\alpha t}\Big(\cos\beta t+\frac{\sigma}{\sqrt{1-2\sigma}}\sin\beta t\Big). \quad (2.8)$$

The moment of onset of the crack front t_* for the step is found from the first equation of (2.8), in which βt_* is denoted by ξ_*

$$e^{-\sqrt{1-2\sigma}\xi_*}(\cos\xi_*-\sqrt{1-2\sigma}\sin\xi_*)=\frac{1-\sigma}{1+\sigma}. \quad (2.9)$$

It can be seen that ξ_* or the dimensionless time $t_{1*} = t_*c/r_0$ for the step is a function of σ only. Knowing t_*, it is possible to calculate all the required initial conditions for the system (2.6).

The solution of the system (2.6) is very complicated and, to all appearances, can be obtained only numerically. An approximate solution in the form of an expansion near the initial point of the front—useful also in the numerical solution—can be obtained by determining the values of the functions and their derivatives at this point.

Differentiating the first three equations in (2.6) with respect to t along R(t) and solving them for $R^{\cdot}$, we get

$$R^{\cdot}=\frac{\sigma\psi^{\cdot\cdot\cdot}/c^2-(1-2\sigma)v}{\sigma\psi^{\cdot\cdot\cdot}/c^3+(1-2\sigma)\sigma_r/E}=$$

$$=\frac{\sigma(f_1^{\cdot\cdot}-f_2^{\cdot\cdot})/c_0-(1-\sigma)v}{\sigma(f_1^{\cdot\cdot}+f_2^{\cdot\cdot})/c_0^2+(1-\sigma)\sigma_r/E}=$$

$$=\frac{\psi^{\cdot\cdot\cdot}/c-(1-2\sigma)c^2\sigma_r/E-(f_1^{\cdot\cdot}+f_2^{\cdot\cdot})}{\psi^{\cdot\cdot\cdot}/c^2+v-(f_1^{\cdot\cdot}-f_2^{\cdot\cdot})/c_0}. \quad (2.10)$$

At the initial point of the front, v and σ_r/E are known; hence the three relations in (2.10), together with the relation derived from differentiation of the last equation in (2.6) with respect to t along the cavity boundary ($r = r_0$),

$$\frac{f_{10}^{\cdot\cdot}-f_{20}^{\cdot\cdot}}{c_0r_0}+\frac{f_1^{\cdot}+f_2^{\cdot}}{r_0^2}=\frac{p^{\cdot}}{E},$$

$$\text{where}\quad \frac{f_{10}^{\cdot\cdot}-f_{20}^{\cdot\cdot}}{c_0}=\frac{p^{\cdot}r_0}{E}-v_0 \quad (2.11)$$

make it possible to calculate $R^{\cdot}_*$, $\psi^{\cdot\cdot\cdot}_*$, $f^{\cdot\cdot}_{1*}+f^{\cdot\cdot}_{2*}$ and $f^{\cdot\cdot}_{1*}-f^{\cdot\cdot}_{2*}$ at the initial point (it is readily seen that $\psi^{\cdot\cdot\cdot}$ has a discontinuity on the characteristic).

We introduce the notation

$$z=-\frac{\sigma}{(1-2\sigma)}\frac{\varphi^{\cdot\cdot\cdot}}{c^2v},\quad y=-\frac{(f_1^{\cdot\cdot}+f_2^{\cdot\cdot})c}{c_0^2v},$$

$$m=-\frac{f_1^{\cdot\cdot}-f_2^{\cdot\cdot}}{c_0v},\qquad N=-\frac{\sigma_r}{E}\frac{c}{v},\quad x=\frac{R^{\cdot}}{c}. \quad (2.12)$$

Then, for (2.10) we get

$$x=\frac{1+z}{N+z}=\frac{1-\sigma+\sigma m}{(1-\sigma)N+\sigma y}=$$

$$=\frac{N+y(1+\sigma)/(1-\sigma)-z/\sigma}{(1+m)/(1-2\sigma)-z/\sigma}. \quad (2.13)$$

It can be seen from (2.11) that

$$m_0=1-\frac{p^{\cdot}r_0}{Ev_0}. \quad (2.14)$$

At the initial point of the front (r_0, t_*), we have enough relations to determine the values of all the functions. For $x_* = R^{\cdot}_*/c$ we get the quadratic equation

$$\Big[N_*+\frac{\sigma(1+m_*)}{1-2\sigma}\Big]x_*^2+(N_*-1)x_*+ \quad (2.15)$$

$$+(1+\sigma)\left(1+\frac{\sigma m_*}{1-\sigma}\right)=0\,. \qquad (2.15)\ \text{(cont'd)}$$

For z_* and y_* we get the expressions

$$z_* = \frac{N_* x_* - 1}{1-x_*}\,,$$
$$y_* = \frac{(1-\sigma)/\sigma + m_*}{x_*} - \frac{(1-\sigma)}{\sigma}N_*\,. \qquad (2.16)$$

Repeating the differentiation of the system of equations, we can find the following derivatives of the unknown functions at the initial point. Differentiation of (2.13), which is more convenient to perform with respect to the dimensionless time $t_1 = tc/r_0$ yields (where the dot denotes differentiation with respect to t_1)

$$x^{\cdot} = \frac{z^{\cdot}(1-x)^2 - N^{\cdot}x(1-x)}{N-1} =$$
$$= \frac{\sigma m^{\cdot} x - x^2[(1-\sigma)N^{\cdot} + \sigma y^{\cdot}]}{1-\sigma+\sigma m} =$$
$$= \frac{\sigma N^{\cdot} + y^{\cdot}\sigma(1+\sigma)/(1-\sigma) - x m^{\cdot}\sigma/(1-2\sigma) - z^{\cdot}(1-x)}{(1+m)\sigma/(1-2\sigma) - z}\,, \qquad (2.17)$$

where

$$N^{\cdot} = -\frac{r_0}{R}\left[xNm + (1-x)z/\sigma - Nyc_0^2/c^2\right], \qquad (2.18)$$

and $m_*^{\cdot}$ and $y_*^{\cdot}$ are related as follows (here we use the conditions at the cavity wall):

$$y_*^{\cdot} = -m_*^{\cdot}/x_* + \beta_1\,,$$
$$\beta_1 = -\left(\frac{m_*}{x_*} + y_*\right)\left[x_*(m_*-1) - y_*\frac{c_0^2}{c^2}\right] -$$
$$-\frac{1}{x_*}\left(1 - \frac{c^2}{c_0^2}x_*^2\right)\left(\frac{c_0^2}{c^2}y_* + \frac{p_*^{\cdot\cdot}}{E}\frac{c}{v_*}\right). \qquad (2.19)$$

Here $p^{\cdot\cdot}$ denotes the second derivative with respect to t_1.

Hence the unknown functions $z_*^{\cdot}$ and $m_*^{\cdot}$ are defined by the system

$$a_1 z_*^{\cdot}(1-x_*) - b_1 = a_2 m_*^{\cdot} - b_2 =$$
$$= -a_3 m_*^{\cdot} - a_4 z_*^{\cdot}(1-x_*) + b_3 \qquad (2.20)$$

where

$$a_1 = \frac{1-x_*}{N_*-1}\,, \quad b_1 = a_1 N_*^{\cdot} x_*\,, \quad a_2 = \frac{2x_*}{(1-\sigma)/\sigma + m_*}\,,$$
$$b_2 = \frac{1}{2}a_2 x_*\left[N_*^{\cdot}(1-\sigma)/\sigma + \beta_1\right],$$
$$a_3 = a_4\left[\frac{\sigma x_*}{1-2\sigma} + \frac{\sigma(1+\sigma)}{(1-\sigma)}\frac{1}{x_*}\right], \qquad (2.21)$$
$$a_4 = \frac{1}{(m_*+1)\sigma/(1-2\sigma) - z_*}\,,$$
$$b_3 = a_4\sigma\left[N_*^{\cdot} + \frac{(1+\sigma)\beta_1}{1-\sigma}\right].$$

The solution of (2.20) has the form

$$z_*^{\cdot}(1-x_*) = \frac{a_2(b_1+b_3) + a_3(b_1-b_2)}{a_1(a_2+a_3) + a_2 a_4}\,, \qquad (2.22)$$

$$m_*^{\cdot} = \frac{a_1(b_3+b_2) - a_4(b_1-b_2)}{a_1(a_2+a_3) + a_2 a_4}\,. \qquad (2.22)\ \text{(cont'd)}$$

Equation (2.17) can be used to calculate $x_*^{\cdot}$. Computation of $R_*^{\cdot}$ and $R_*^{\cdot\cdot}$, yields the expansion for $\Delta R_1 = R/r_0 - 1$ as follows:

$$\Delta R_1 = x_*\Delta t_1 + {}^1/_2\, x_*^{\cdot}(\Delta t_1)^2 + \ldots, \quad \Delta t_1 = t_1 - t_{1*}\,. \qquad (2.23)$$

An analogous expansion can be also obtained for the rate of motion of the cavity wall:

$$v_0 = v_* + v_{0*}^{\cdot}\Delta t_1 + \frac{1}{2}v_{0*}^{\cdot\cdot}(\Delta t_1)^2 + \ldots$$

The notation $v_{0*}^{\cdot}$ and $v_{0*}^{\cdot\cdot}$ employed in this expression denotes the differentiation of v_0 with respect to time. The derivatives of v_0 are expressed by $\psi_*^{\cdot\cdot}$, $\psi_*^{\cdot\cdot\cdot}$. and $\psi_*^{\cdot\cdot\cdot\cdot}$, which have the following form:

$$\psi_*^{\cdot\cdot} = (1-2\sigma)\,cr_0 N_* v_*\,, \quad \psi_*^{\cdot\cdot\cdot} = -c^2 v_* z_*(1-2\sigma)/\sigma\,,$$
$$\psi_*^{\cdot\cdot\cdot\cdot} = -\frac{(1-2\sigma)}{\sigma}\frac{v_* c^3}{r_0(1-x_*)}\{z_*^{\cdot} -$$
$$- z_*[y_* c_0^2/c^2 + x_*(1-m_*)]\}\,. \qquad (2.24)$$

Then,

$$v_{0*}^{\cdot}/v_* = (1-2\sigma)(N_* - z/\sigma),$$
$$v_{0*}^{\cdot\cdot}/v_* = -\frac{1-2\sigma}{\sigma}\Big\{z_* +$$
$$+\frac{z_*^{\cdot} - z_*[y_* c_0^2/c^2 + x_*(1-m_*)]}{1-x_*}\Big\}\,. \qquad (2.25)$$

From (1.11) it follows that $\psi^{\cdot\cdot}$ is continuous across the characteristic that originates at the point (r_0, t_*). Using the second formula in (2.24), it is readily shown that $\psi^{\cdot\cdot\cdot}$ (as already mentioned) has a discontinuity at this characteristic.

§3. We will examine the simple but important case in which the pressure at the cavity wall is given in the form of a step function: at a certain moment (denoted by $t = 0$), there develops a pressure p_0 which thereafter remains constant. For an elastic solution and the determination of t_* we have derived formulas (2.7) through (2.9). The condition

$$p = p_0 = \text{const} \qquad (3.1)$$

leads to a simplification of the formulas. Thus,

$$N_* = \frac{p_0 c}{E v_*} = \frac{(1-\sigma)}{(1+\sigma)(1-2\sigma)}\,\frac{\exp(\sqrt{1-2\sigma}\,\xi_*)}{[\cos\xi_* + (\sigma/\sqrt{1-2\sigma})\sin\xi_*]}\,,$$

and since ξ_* is a function only of Poisson's ratio σ, N_* is also a function only of σ. If the condition (3.1) is satisfied, $m_* = 1$ and, as can be seen from (2.15) and (2.16), x_*, z_*, and y_* are also functions only of σ. Hence, the initial propagation rate of the front $R_*^{\cdot}$ is a function only of σ and c, both of which are characteristics of the medium. Table 1 and the following compilation give the results of the calculations for various values of Poisson's ratio.

Table 1

σ	ξ_*	N_*	x_*	z_*	y_*	$\dot{z}_*(1-x_*)$	$\dot{m}_*$	$\dot{x}_*$
0.10	0.1061	1.1175	0.9033	0.0990	1.0117	0.0087	0.9037	0.0069
0.25	0.3031	1.4027	0.7533	0.2294	1.1021	0.0422	0.7621	0.0235
0.35	0.4860	1.7709	0.6236	0.2773	1.2927	0.4253	1.1727	0.2004

σ	$\dot{f}_{1*}/r_0v_*$	$\dot{f}_{2*}/\dot{f}_{1*}$	$\ddot{f}_{1*}/c_0v_*$	$\ddot{f}_{2*}/\ddot{f}_{1*}$
0.10	1.0525	—0.0499	—1.0002	0.0002
0.25	1.1402	—0.1230	—1.0030	0.0030
0.35	1.1989	—0.1659	—1.0102	0.0101

In addition to the formulas derived above, we will give several more:

$$\frac{f_{1*}^{\cdot}}{r_0v_*} = \frac{1}{2}\left(1+\frac{c_0}{c}N_*\right), \qquad \frac{f_{2*}^{\cdot}}{r_0v_*} = 1-\frac{f_{1*}^{\cdot}}{r_0v_*},$$

$$\frac{f_{1*}^{\cdot\cdot}}{c_0v_*} = -\frac{1}{2}\left(1+\frac{c_0}{c}y_*\right), \qquad \frac{f_{2*}^{\cdot\cdot}}{c_0v_*} = 1+\frac{f_{1*}^{\cdot\cdot}}{c_0v_*},$$

$$\frac{f_{1*}^{\cdot\cdot\cdot}r_0}{v_*c_0c} =$$

$$= -\frac{m_*^{\cdot}+y_*^{\cdot}c_0/c+(m_*+y_*c_0/c)[x_*(m_*-1)-y_*c_0^2/c^2]}{2(1-x_*c_0/c)},$$

$$\frac{f_{2*}^{\cdot\cdot\cdot}r_0}{v_*c_0c} =$$

$$= \frac{m_*^{\cdot}-y_*^{\cdot}c_0/c+(m_*-y_*c_0/c)[x_*(m_*-1)-y_*c_0^2/c^2]}{2(1+x_*c_0/c)}. \tag{3.3}$$

Table 1 also gives the expansion coefficients for ΔR_1. If there were no crack formation and normal elastic motion continued, then v_0/v_* would be expressed by

$$\frac{v_0}{v_*} = \exp[-\alpha(t-t_*)]\,\frac{\cos\beta t+(\sigma/\sqrt{1-2\sigma})\sin\beta t}{\cos\beta t_*+(\sigma/\sqrt{1-2\sigma})\sin\beta t_*}. \tag{3.4}$$

Computation of the expansion (3.4) near t_* shows that the presence of a crack zone tends to slow down the rate of motion of the cavity wall. In the formation of the crack front at the moment t_*, the rate of motion of the wall is continuous, but were its first derivative experiences a discontinuity.

§4. It can be seen from the computations that the derivatives of the function f_2 are very small compared to those of f_1. Hence an approximate solution may be proposed in which the derivatives of f_2 are neglected. Then, from the fourth equation in (2.6) we obtain an equation for f_1:

$$\frac{df_1(t)}{dt}+\frac{c_0}{r_0}f_1(t) = \frac{pc_0r_0}{E}. \tag{4.1}$$

For $p = p_0 = \text{const}$, the solution has the form

$$f_1(t) = p_0r_0^2/E+K\exp(-c_0t/r_0),$$

where K is a constant of integration that can be expressed in terms of the rate of propagation at the point (r_0, t_*), which in our approximation is $v_* = \dot{f}_{1*}/r_0$. From the second and fourth equations of (2.6), it also follows that

$$\frac{v_*}{c_0} = \frac{f_{1*}^{\cdot}}{c_0r_0} = \frac{(1-\sigma)}{\sigma}\frac{f_{1*}}{r_0^2}, \qquad \frac{f_{1*}^{\cdot}}{c_0r_0}+\frac{f_{1*}}{r_0^2} = \frac{p_0}{E}.$$

Hence we get the following final expression for $f_1(t)$:

$$f_1(t) = \frac{v_*r_0^2}{c_0}\left\{\frac{1}{1-\sigma}-\exp\left[-\frac{c_0(t-t_*)}{r_0}\right]\right\}. \tag{4.2}$$

With the aid of the second equation of (2.6), it is now possible to obtain the equation of motion of the crack front

$$\Delta t_1 = \frac{c}{c_0}[\Delta R_1+\ln(1+\sigma\Delta R_1)]. \tag{4.3}$$

This function is given in Fig. 2 (the broken curve corresponds to x_* from Table 1). Unfortunately, it is not yet possible to compare this solution with the exact solution (2.6), but it is possible to compare the values of the various functions at the initial point of the front. In the first place, for the point of formation of the front itself we obtain the expression

$$t_* = \operatorname{arc\,tg}\frac{\sqrt{1-\sigma^2}-\sqrt{1-2\sigma}}{\sqrt{(1-\sigma^2)(1-2\sigma)}+\sigma}. \tag{4.4}$$

This expression can be used as a first approximation in the solution of the transcendental equation (2.18). For the parameters at the initial point we get

$$N_* = \frac{1}{(1-\sigma)}\frac{c}{c_0}, \qquad x_* = \frac{1}{(1+\sigma)}\frac{c}{c_0},$$
$$x_*^{\cdot} = \frac{\sigma^2}{(1+\sigma)^3}\frac{c_0^2}{c^2}. \tag{4.5}$$

Table 2 gives the values of these parameters for various σ. A comparison of the values in Tables 1 and 2 confirms the applicability of the approximate method, in particular, for media with a small Poisson ratio. Another peculiarity of the motion of the crack front, which can be seen from Table 1, should be noted. At the very beginning, the crack front moves with an acceleration $\dot{x}_* > 0$. This is also apparent from the approximate solution, the first and second derivatives of which have the form

$$\frac{d(\Delta t_1)}{d(\Delta R_1)} = \frac{c}{c_0}\left(1+\frac{\sigma}{1+\sigma\Delta R_1}\right), \quad \frac{d^2(\Delta t_1)}{d(\Delta R_1)^2} = -\frac{c}{c_0}\frac{\sigma^2}{(1+\sigma\Delta R_1)^2}.$$

With increasing R, the acceleration decreases, while $R^{\cdot}$ tends toward c.

Note that an analogous approximate solution may be obtained for $\sigma_{\theta *} \neq 0$. We introduce the notation $\delta = \sigma_{\theta *}/p_0$. Then the motion of the front is described by the equation

$$\Delta t_1 = \frac{c}{c_0}\left[\Delta R_1+\ln\frac{(1-\delta)(1+\sigma\Delta R_1)}{1-\delta(1+\Delta R_1)^2}\right]. \tag{4.6}$$

It can be seen that Δt_1 becomes ∞ at $\delta(1+\Delta R_1)^2 = 1$; hence $R_\infty = r_0\sqrt{\delta}$. This expression is a factor $\sqrt{2}$ greater than that obtained in [1]. For the derivatives $d(\Delta t_1/d(\Delta R_1)$ and $d^2(\Delta t_1)/d(\Delta R_1)^2$ we get the expression

$$\frac{d(\Delta t_1)}{d(\Delta R_1)} = \frac{c}{c_0}\left[1+\frac{\sigma}{1+\sigma\Delta R_1}+\frac{2\delta(1+\Delta R_1)}{1-\delta(1+\Delta R_1)^2}\right],$$

$$\frac{d^2(\Delta t_1)}{d(\Delta R_1)^2} = \frac{c}{c_0}\left\{-\frac{\sigma^2}{(1+\sigma\Delta R_1)^2}+2\delta\frac{1+\delta(1+\Delta R_1)^2}{[1-\delta(1+\Delta R_1)^2]^3}\right\}.$$

For the initial point of the front and certain quantities at this point we have the expressions

$$\xi_* = \operatorname{arc\,tg} \frac{(1-\delta)\sqrt{1-\sigma^2} - (1-2\delta)\sqrt{1-2\sigma}}{(1-2\delta)[\sigma + \sqrt{(1-\sigma^2)(1-2\sigma)}]},$$

$$N_* = \frac{1}{(1-\sigma)(1-\delta)} \frac{c}{c_0}, \quad x_* = \frac{c_0/c}{1+\sigma+2\delta/(1-\delta)},$$

$$x_*^{\cdot} = -\frac{c_0^2}{c^2} \frac{[-\sigma^2 + 2\delta(1+\delta)/(1-\delta)^2]}{[1+\sigma+2\delta/(1-\delta)]^3},$$

$$\left[\frac{d(\Delta t_1)}{d(\Delta R_1)}\right]_* = \frac{1}{x_*},$$

$$\left[\frac{d^2(\Delta t_1)}{d(\Delta R_1)^2}\right]_* = \frac{c}{c_0}\left[-\sigma^2 + 2\delta\frac{(1+\delta)}{(1-\delta)^2}\right].$$

Quantity $[d^2(\Delta t_1)5d(\Delta R_1)^2]_*$ becomes positive for $\sigma = 0.1$, beginning with $\delta = 0.005$, for $\sigma = 0.25$ beginning with $\delta = 0.0283$, and for $\sigma = 0.35$ beginning with $\delta = 0.0534$. Thus, the presence of a certain $\sigma_{\theta *}$ leads to deceleration and halting of the crack front.

Table 2

σ	x_*	$x_*^{\cdot}$	N_*
0.1	0.9	0.00817	1.12
0.25	0.73	0.0366	1.46
0.35	0.585	0.053	1.95

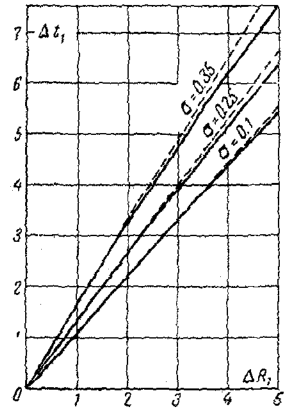

Fig. 2

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